# Arithmetic Sequences

A simple way to generate a sequence is to start with a number $a$, and add to it a fixed constant $d$, over and over again. This type of sequence is called an arithmetic sequence.

**Definition:** An **arithmetic sequence** is a sequence of the form

$$a, a + d, a + 2d, a + 3d, a + 4d, \ldots$$

The number $a$ is the **first term**, and $d$ is the **common difference** of the sequence. The ***n*th term** of an arithmetic sequence is given by

$$a_n = a + (n - 1)d$$

The number $d$ is called the common difference because any two consecutive terms of an arithmetic sequence differ by $d$, and it is found by subtracting any pair of terms $a_n$ and $a_{n+1}$. That is

$$d = a_{n+1} - a_n$$

**Is the Sequence Arithmetic?**

**Example 1:** Determine whether or not the sequence is arithmetic. If it is arithmetic, find the common difference.

(a) 2, 5, 8, 11, …
(b) 1, 2, 3, 5, 8, …

**Solution (a):** In order for a sequence to be arithmetic, the differences between each pair of adjacent terms should be the same. If the differences are all the same, then $d$, the common difference, is that value.

**Step 1:** First, calculate the difference between each pair of adjacent terms.

$5 - 2 = 3$
$8 - 5 = 3$
$11 - 8 = 3$

**Step 2:** Now, compare the differences. Since each pair of adjacent terms has the same difference 3, the sequence is arithmetic and the common difference $d = 3$.

**Example 1 (Continued):**

**Solution (b):**

**Step 1:** Calculate the difference between each pair of adjacent terms.

$2 - 1 = 1$
$3 - 2 = 1$
$5 - 3 = 2$
$8 - 5 = 3$

**Step 2:** Compare the differences. Since the differences between each pair of adjacent terms are not all the same, the sequence is not arithmetic.

An arithmetic sequence is determined completely by the first term $a$, and the common difference $d$. Thus, if we know the first two terms of an arithmetic sequence, then we can find the equation for the $n$th term.

**Finding the Terms of an Arithmetic Sequence:**

**Example 2:** Find the $n$th term, the fifth term, and the $100^{th}$ term, of the arithmetic sequence determined by $a = 2$ and $d = 3$.

**Solution:** To find a specific term of an arithmetic sequence, we use the formula for finding the $n$th term.

**Step 1:** The $n$th term of an arithmetic sequence is given by

$$a_n = a + (n - 1)d.$$

So, to find the $n$th term, substitute the given values $a = 2$ and $d = 3$ into the formula.

$$a_n = 2 + (n - 1)3$$

**Step 2:** Now, to find the fifth term, substitute $n = 5$ into the equation for the $n$th term.

$$\begin{aligned} a_5 &= 2 + (5 - 1)3 \\ &= 14 \end{aligned}$$

**Step 3:** Finally, find the $100^{th}$ term in the same way as the fifth term.

$$\begin{aligned} a_{100} &= 2 + (100 - 1)3 \\ &= 299 \end{aligned}$$

**Example 3:** Find the common difference, the fifth term, the *n*th term, and the 100th term of the arithmetic sequence.

(a) 4, 14, 24, 34, …

(b) $t+3, t+\frac{15}{4}, t+\frac{9}{2}, t+\frac{21}{4}, ...$

**Solution (a):** In order to find the *n*th and 100th terms, we will first have to determine what $a$ and $d$ are. We will then use the formula for finding the *n*th term.

**Step 1:** First, we will determine what $a$ and $d$ are. The number $a$ is always the first term of the sequence, so

$$a = 4$$

The difference between any pair of adjacent terms should be the same because the sequence is arithmetic, so we can choose any one pair to find the common difference $d$. If we choose the first two terms then

$$\begin{aligned} d &= 14 - 4 \\ &= 10 \end{aligned}$$

**Step 2:** Since we are given the fourth term, we can add the common difference $d = 10$ to it to get the fifth term.

$$\begin{aligned} a_5 &= 34 + 10 \\ &= 44 \end{aligned}$$

**Step 3:** Now to find the *n*th term, substitute $a = 4$ and $d = 10$ into the formula for the *n*th term.

$$a_n = 4 + (n - 1)10$$

**Step 4:** Finally, substitute $n = 100$ into the equation for the *n*th term to get the 100th term.

$$\begin{aligned} a_{100} &= 4 + (100 - 1)10 \\ &= 994 \end{aligned}$$

**Example 3 (Continued):**

**Solution (b):**

**Step 1:** Calculate *a* and *d*.

$$a = t + 3$$

$$\begin{aligned} d &= \left(t + \frac{15}{4}\right) - (t + 3) \\ &= t + \frac{15}{4} - t - 3 \\ &= \frac{15}{4} - 3 \\ &= \frac{3}{2} \end{aligned}$$

**Step 2:** The fifth term is the fourth term plus the common difference. Therefore,

$$\begin{aligned} a_5 &= \left(t + \frac{21}{4}\right) + \frac{3}{2} \\ &= t + \frac{24}{4} \\ &= t + 6 \end{aligned}$$

**Step 3:** Now, substitute $a = t + 3, d = \frac{3}{2}$ into the formula for the *n*th term.

$$a_n = (t + 3) + (n - 1)\frac{3}{2}$$

**Step 4:** Finally, substitute *n* = 100 into the equation for the *n*th term that we just found.

$$\begin{aligned} a_n &= (t + 3) + (100 - 1)\frac{3}{2} \\ &= t + 3 + (99)\frac{3}{2} \\ &= t + \frac{303}{2} \end{aligned}$$

**Partial Sums of an Arithmetic Sequence:**

To find a formula for the sum, $S_n$, of the first $n$ terms of an arithmetic sequence, we can write out the terms as

$$S_n = a + (a+d) + (a+2d) + ... + [a + (n-1)d].$$

This same sum can be written in reverse as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + ... + [a_n - (n-1)d]$$

Now, add the corresponding terms of these two expressions for $S_n$ to get

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + ... + [a + (n-1)d] \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + ... + [a_n - (n-1)d] \\ \hline 2S_n &= (a + a_n) + (a + a_n) + (a + a_n) + ... + (a + a_n) \end{aligned}$$

The right hand side of this expression contains $n$ terms, each equal to $a + a_n$, so

$$\begin{aligned} 2S_n &= n(a + a_n) \\ S_n &= \frac{n}{2}(a + a_n). \end{aligned}$$

**Definition:** For the arithmetic sequence $a_n = a + (n-1)d$, the ***n*th partial sum**

$$S_n = a + (a+d) + (a+2d) + (a+3d) + ... + [a + (n-1)d]$$

is given by either of the following formulas.

**1.** $S_n = \frac{n}{2}[2a + (n-1)d]$

**2.** $S_n = n\left(\frac{a + a_n}{2}\right)$

The *n*th partial sum of an arithmetic sequence can also be written using summation notation.

$$\sum_{i=1}^{n} ki - c$$

represents the sum of the first *n* terms of an arithmetic sequence having the first term $a = k(1) + c = k + c$ and the *n*th term $a_n = k(n) + c = kn + c$. We can find this sum with the second formula for $S_n$ given above.

**Example 4:** Find the partial sum $S_n$ of the arithmetic sequence that satisfies the given conditions.

(a) $a = 6$, $d = 3$, and $n = 7$

(b) $\sum_{i=1}^{14} 2i - 7$

**Solution (a):** To find the *n*th partial sum of an arithmetic sequence, we can use either of the formulas

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \text{ or } S_n = n\left(\frac{a + a_n}{2}\right)$$

**Step 1:** To use the first formula for the *n*th partial sum, we only need to substitute the given values $a = 6$, $d = 3$, and $n = 7$ into the equation.

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_7 = \frac{7}{2}\left[2(6) + (7-1)3\right]$$

$$= \frac{7}{2}[12 + 18]$$

$$= 105$$

**Example 4 (Continued):**

**Solution (b):** This is the sum of the first fourteen terms of the arithmetic sequence having $a_n = 2n - 7$.

**Step 1:** Since the partial sum is given in summation notation, we must first find $a$ and $a_n$. From the given information we know $k = 2$, $c = -7$, and $n = 14$, so

$$\begin{aligned} a &= k + c \\ &= 2 + (-7) \\ &= -5 \end{aligned}$$

$$\begin{aligned} a_n &= kn + c \\ a_{14} &= 2(14) + (-7) \\ &= 21 \end{aligned}$$

**Step 2:** Now that we know $a = -5$, $n = 14$, and $a_{14} = 21$, we can substitute these values into the second formula for the $n$th partial sum to find the fourteenth partial sum.

$$\begin{aligned} S_{14} &= n\left(\frac{a + a_{14}}{2}\right) \\ &= 14\left(\frac{-5 + 21}{2}\right) \\ &= 112 \end{aligned}$$

**Example 5:** Find the sum of the first 37 even numbers.

**Solution:**

**Step 1:** First, we must find the values $a$, $d$, and $n$. Since the first even number is zero, $a = 0$. The next even number is 2, so $d = 2 - 0 = 2$. Since we are told to find the sum of the first 37 even numbers, $n = 37$.

**Example 5 (Continued):**

**Step 2:** Now that we know $a = 0$, $d = 2$, and $n = 37$ we can solve this problem the same way as in the previous example. First find $a_{37}$, and then substitute the values for $a$, $d$, and $a_{37}$ into the equation for the $n$th partial sum. Thus,

$$\begin{aligned} a_{37} &= 0 + (37 - 1)2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} S_{37} &= 37\left(\frac{0+18}{2}\right) \\ &= 363 \end{aligned}$$

**Example 6:** A partial sum of an arithmetic sequence is given. Find the sum.

$$1 + 8 + 15 + \ldots + 78$$

**Solution:**

**Step 1:** As in the previous example, we must first find $a$, $d$, and $n$. The values $a$ and $d$ are easy to find.

$$a = 1$$

$$\begin{aligned} d &= 8 - 1 \\ &= 7 \end{aligned}$$

Now, finding $n$ is a bit more work because we are not explicitly told how many numbers we will be summing. We know $a$ and $d$, and we know the $n$th term, so we will substitute these values into the formula for the $n$th term of a sequence.

$$\begin{aligned} a_n &= a + (n-1)d \\ 78 &= 1 + (n-1)7 \end{aligned}$$

Now solve for $n$.

$$\begin{aligned} 77 &= (n-1)7 \\ 11 &= n - 1 \\ 12 &= n \end{aligned}$$

Therefore, we will be summing twelve terms and $78 = a_{12}$.

**Example 6 (Continued):**

**Step 2:** Now that we know $a = 1$, $n = 12$, and $a_{12} = 78$ we can solve this problem the same way as in example 4. Substitute the values for $a$, $d$, and $a_{12}$ into the formula for the $n$th partial sum.

$$\begin{aligned} S_{12} &= 12\left(\frac{1+78}{2}\right) \\ &= 474 \end{aligned}$$