Geometric Sequences

Another simple way of generating a sequence is to start with a number "*a*" and repeatedly multiply it by a fixed nonzero constant "*r*". This type of sequence is called a geometric sequence.

Definition: A **geometric sequence** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

The number *a* is the **first term**, and *r* is the **common ratio** of the sequence. The ***n*th term** of a geometric sequence is given by

$$a_n = ar^{n-1}.$$

The number *r* is called the common ratio because any two consecutive terms of the sequence differ by a multiple of *r*, and it is found by dividing any term a_{n+1} after the first by the preceding term a_n. That is

$$r = \frac{a_{n+1}}{a_n}.$$

Is the Sequence Geometric?

Example 1: Determine whether the sequence is geometric. If it is geometric, find the common ratio.

(a) $2, 8, 32, 128, \ldots$
(b) $1, 2, 3, 5, 8, \ldots$

Solution (a): In order for a sequence to be geometric, the ratio of any term to the one that precedes it should be the same for all terms. If they are all the same, then *r*, the common difference, is that value.

Step 1: First, calculate the ratios between each term and the one that precedes it.

$$\frac{8}{2} = 4$$
$$\frac{32}{8} = 4$$
$$\frac{128}{32} = 4$$

Example 1 (Continued):

Step 2: Now, compare the ratios. Since the ratio between each term and the one that precedes it is 4 for all the terms, the sequence is geometric, and the common ratio $r = 4$.

Solution (b):

Step 1: Calculate the ratios between each term and the one that precedes it.

$$\frac{2}{1} = 1$$
$$\frac{3}{2} = \frac{3}{2}$$
$$\frac{5}{3} = \frac{5}{3}$$
$$\frac{8}{5} = \frac{8}{5}$$

Step 2: Compare the ratios. Since they are not all the same, the sequence is not geometric.

Similar to an arithmetic sequence, a geometric sequence is determined completely by the first term *a*, and the common ratio *r*. Thus, if we know the first two terms of a geometric sequence, then we can find the equation for the *n*th term.

Finding the Terms of a Geometric Sequence:

Example 2: Find the *n*th term, the fifth term, and the 100th term, of the geometric sequence determined by $a = 6, r = \frac{1}{3}$.

Solution: To find a specific term of a geometric sequence, we use the formula for finding the *n*th term.

Step 1: The *n*th term of a geometric sequence is given by

$$a_n = ar^{n-1}$$

So, to find the *n*th term, substitute the given values $a = 6, r = \frac{1}{3}$ into the formula.

$$a_n = 6\left(\frac{1}{3}\right)^{n-1}$$

Example 2 (Continued):

Step 2: Now, to find the fifth term, substitute $n = 5$ into the equation for the nth term.

$$\begin{aligned} a_5 &= 6\left(\frac{1}{3}\right)^{5-1} \\ &= 6\left(\frac{1}{3^4}\right) \\ &= \frac{6}{81} \\ &= \frac{2}{27} \end{aligned}$$

Step 3: Finally, find the 100th term in the same way as the fifth term.

$$\begin{aligned} a_5 &= 6\left(\frac{1}{3}\right)^{100-1} \\ &= 6\left(\frac{1}{3^{99}}\right) \\ &= \frac{2 \cdot 3}{3^{99}} \\ &= \frac{2}{3^{98}} \end{aligned}$$

Example 3: Find the common ratio, the fifth term and the nth term of the geometric sequence.

(a) $-1, 9, -81, 729, \ldots$

(b) $\frac{1}{2}, \frac{t}{6}, \frac{t^2}{18}, \frac{t^3}{54}, \ldots$

Solution (a): In order to find the nth term, we will first have to determine what a and r are. We will then use the formula for finding the nth term of a geometric sequence.

Example 3 (Continued):

Step 1: First, determine what *a* and *r* are. The number *a* is always the first term of the sequence, so

$$a = -1.$$

The ratio between any term and the one that precedes it should be the same because the sequence is geometric, so we can choose any pair to find the common ratio *r*. If we choose the first two terms

$$\begin{aligned} r &= \frac{9}{-1} \\ &= -9. \end{aligned}$$

Step 2: Since we are given the fourth term, we can multiply it by the common ratio $r = -9$ to get the fifth term.

$$\begin{aligned} a_5 &= a_4 \cdot r \\ &= 729(-9) \\ &= -6561 \end{aligned}$$

Step 3: Now, to find the *n*th term, substitute $a = -1, r = -9$ into the formula for the *n*th term of a geometric sequence.

$$\begin{aligned} a_n &= ar^{n-1} \\ &= (-1)(-9)^{n-1} \\ &= -(-9)^{n-1} \end{aligned}$$

Example 3 (Continued):

Solution (b):

Step 1: Calculate *a* and *r*.

$$
\begin{aligned}
a &= \frac{1}{2} \\
r &= \frac{\left(\frac{t}{6}\right)}{\left(\frac{1}{2}\right)} \\
&= \left(\frac{t}{6}\right)\left(\frac{2}{1}\right) \\
&= \frac{t}{3}
\end{aligned}
$$

Step 2: The fifth term is the fourth term multiplied by the common ratio. Therefore,

$$
\begin{aligned}
a_5 &= a_4 \cdot r \\
&= \left(\frac{t^3}{54}\right)\left(\frac{t}{3}\right) \\
&= \frac{t^4}{162}
\end{aligned}
$$

Step 3: Now, substitute $a = \frac{1}{2}, r = \frac{t}{3}$ into the formula for the *n*th term.

$$a_n = \left(\frac{1}{2}\right)\left(\frac{t}{3}\right)^{n-1}$$

Partial Sums of a Geometric Sequence:

We can start developing a formula for the sum of the first *n* terms of a geometric sequence, S_n, by writing it out in long form.

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

Next, we multiply both sides by *r*.

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n$$

We subtract the first result from the second.

$$\begin{array}{c} S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n \\ \hline rS_n - S_n = (ar - a) + \left(ar^2 - ar\right) + \left(ar^3 - ar^2\right) + \left(ar^4 - ar^3\right) + \ldots + \left(ar^n - ar^{n-1}\right) \end{array}$$

Using the commutative and associative properties to rearrange the terms on the right,

$$rS_n - S_n = (ar - ar) + \left(ar^2 - ar^2\right) + \left(ar^3 - ar^3\right) + \left(ar^4 - ar^4\right) + \ldots + \left(ar^n - a\right)$$

so if $r \neq 1$,

$$S_n(1-r) = a\left(r^n - 1\right)$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

Definition: For the geometric sequence $a_n = ar^{n-1}$, the ***n*th partial sum**

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \quad (r \neq 1)$$

is given by

$$S_n = a\frac{1-r^n}{1-r}$$

Written using summation notation, the *n*th partial sum of a geometric sequence is

$$\sum_{i=1}^{n} k \cdot r^i .$$

This represents the sum of the first *n* terms of a geometric sequence having first term $a = k \cdot r^1 = kr$ and common ratio *r*.

Example 4: Find the partial sum S_n of the geometric sequence that satisfies the given conditions.

(a) $a = 1, r = 2, n = 7$

(b) $\sum_{i=1}^{5}(-8)(-\frac{1}{2})^i$

Solution (a): To find the *n*th partial sum of a geometric sequence, we use the formula derived above.

Step 1: To use the formula for the *n*th partial sum of a geometric sequence, we only need to substitute the given values $a = 1, r = 2, n = 7$ into the formula.

$$S_n = a\frac{1-r^n}{1-r}$$

$$S_7 = (1)\left(\frac{1-2^7}{1-2}\right)$$

$$= \frac{1-128}{-1}$$

$$= 127$$

Solution (b): This is the sum of the first five terms of the geometric sequence with $a_n = 4\left(-\frac{1}{2}\right)^{n-1}$.

Step 1: Since the partial sum is given in summation notation, we must first find *a* and *r*. From the given information, we know $k = -8, r = -\frac{1}{2}, n = 5$. So,

$$r = -\frac{1}{2}$$

$$a = kr$$

$$= (-8)\left(-\frac{1}{2}\right)$$

$$= 4$$

Example 4 (Continued):

Step 2: Now that we know $a = 4, r = -\frac{1}{2}$, we can substitute these values into the formula for the nth partial sum to find the fifth partial sum.

$$\begin{aligned} S_5 &= 4\left(\frac{1-\left(-\frac{1}{2}\right)^5}{1-\left(-\frac{1}{2}\right)}\right) \\ &= 4\left(\frac{1-\left(-\frac{1}{32}\right)}{\frac{3}{2}}\right) \\ &= 4\left(\frac{33}{32}\right)\left(\frac{2}{3}\right) \\ &= \frac{11}{4} \end{aligned}$$

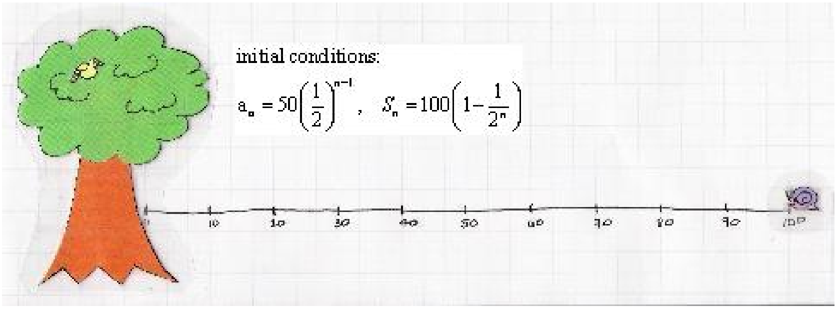
Infinite Series:

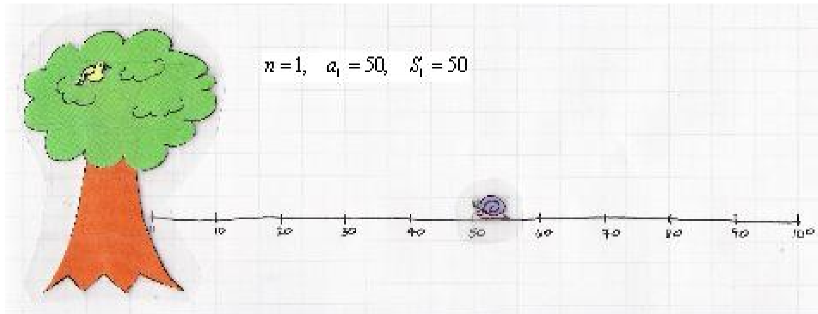
An expression of the form

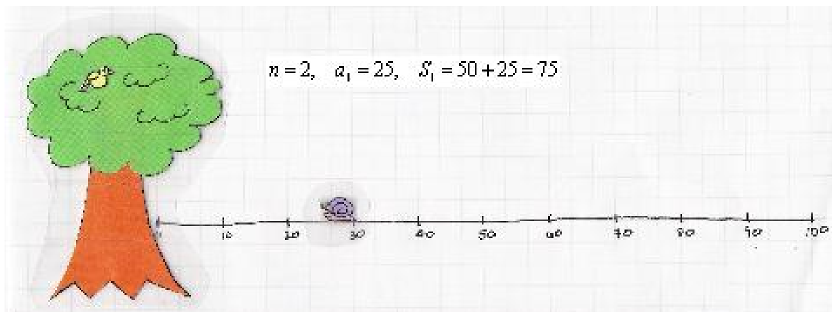
$$a_1 + a_2 + a_3 + a_4 + ...$$

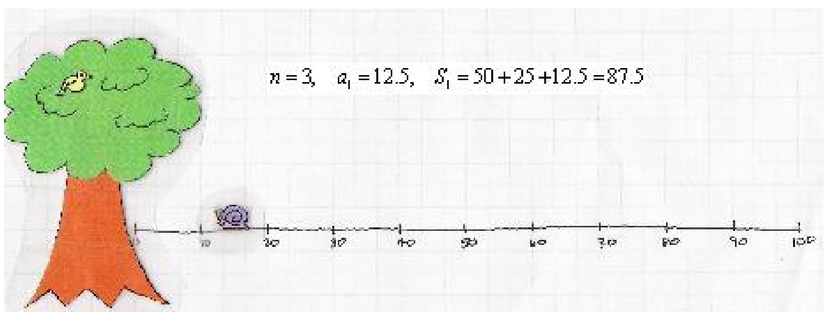
is called an **infinite series**. The dots mean that we are to continue the addition indefinitely. The idea of adding infinitely many numbers and getting a finite number may seem strange, but consider the following scenario.

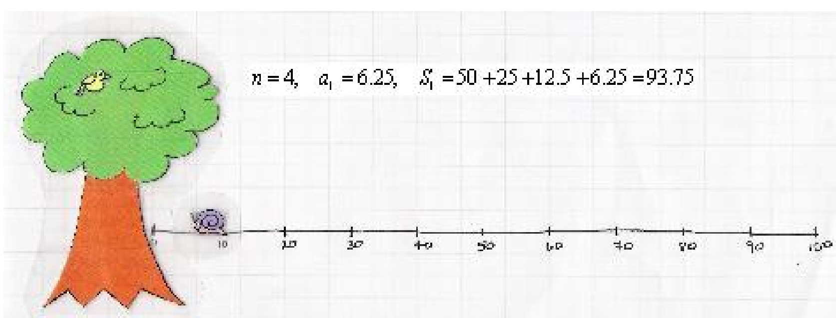
To begin with, a snail is 100 feet from a tree. On the first day, it travels half the distance to the tree. On the second day, it travels half the remaining distance to the tree, and on the third day half of the remaining distance again. This process of traveling half the remaining distance per day can continue indefinitely and at the end of each day some distance will still remain. See the following figures.











Does this mean that the snail will never reach the tree? Of course not. Let's add up the distance our snail has traveled:

$$100\left(\frac{1}{2}\right)+\left(100\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)+\left(\left(100\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\right)+\ldots$$
$$=\frac{100}{2}+\frac{100}{4}+\frac{100}{8}+\frac{100}{16}+\ldots+\frac{100}{2^n}+\ldots$$

This is an infinite series, and we note two things about it: First, no matter how many terms of this series we add, the total will never exceed 100. Second, the more terms of this series we add, the closer the sum is to 100. This suggests that the number 100 can be written as the sum of infinitely smaller numbers:

$$100 = \frac{100}{2} + \frac{100}{4} + \frac{100}{8} + \frac{100}{16} + \ldots + \frac{100}{2^n} + \ldots$$

To make this clearer, let's look at the partial sums of this series:

$$S_1 = \frac{100}{2} = 50$$
$$S_2 = \frac{100}{2} + \frac{100}{4} = 75$$
$$S_3 = \frac{100}{2} + \frac{100}{4} + \frac{100}{8} = 87.5$$
$$S_4 = \frac{100}{2} + \frac{100}{4} + \frac{100}{8} + \frac{100}{16} = 93.75$$

and the general form

$$S_n = 100 - \frac{100}{2^n}.$$

As n gets larger and larger, we are adding more and more of the terms of this series. Intuitively, as n gets larger, S_n gets closer to the sum of the series. Now notice that as n gets large, $1/2^n$ gets closer and closer to zero. Thus S_n gets closer to $100 - 0 = 100$. Using the notation of Section 5.5, we can write

$$S_n \to 100 \quad \text{as} \quad n \to \infty.$$

In general, if S_n gets close to a finite number S as n gets large, we say that S is **the sum of the infinite series**.

Infinite Geometric Series:

An **infinite geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots \; .$$

We learned earlier in this section that the sum S_n of the first *n* terms of a geometric sequence is given by

$$S_n = a\frac{1-r^n}{1-r} \quad (r \neq 1).$$

If $|r|<1$, that is, if $-1<r<1$, the term r^n steadily decreases in absolute value as *n* increases, getting closer and closer to zero. The fact that r^n gets nearer and nearer to zero as *n* takes on larger and larger values suggests that the sums S_n should themselves be getting closer and closer to some certain value *S*. This is actually what happens, and we write

$$S_n \to \frac{a}{1-r} \quad \text{as} \quad n \to \infty.$$

If $|r|>1$ and $a \neq 0$, the terms ar^n increase in absolute value without bound as *n* increases without bound, and the sums S_n do also. If $|r|=1$ and $a \neq 0$,

$$S_n \text{ does not exist} \quad \text{as} \quad n \to \infty.$$

Definition: If $|r|<1$, then the **sum of the infinite geometric series**

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots$$

is

$$S = \frac{a}{1-r}.$$

Example 5: Find the sum of the infinite geometric series.

$$4-\frac{4}{3}+\frac{4}{9}-\frac{4}{27}+\ldots$$

Solution: To find the sum of an infinite geometric series, we use the formula

$$S=\frac{a}{1-r}$$

Step 1: First, we must find *a* and *r*. This is done in the same way as for a geometric sequence. The value *a* is always the first term in the series, and the common ratio *r* is the ratio between any term and the one preceding it. So,

$$a=4$$

$$\begin{aligned} r &= \frac{\left(-\frac{4}{3}\right)}{4} \\ &= \left(-\frac{4}{3}\right)\left(\frac{4}{1}\right) \\ &= -\frac{1}{3} \end{aligned}$$

Step 2: Now substitute the values $a=4, r=-\frac{1}{3}$ into the formula for the sum of an infinite geometric series.

$$\begin{aligned} S &= \frac{4}{1-\left(-\frac{1}{3}\right)} \\ &= \frac{4}{\left(\frac{4}{3}\right)} \\ &= \left(\frac{4}{1}\right)\left(\frac{3}{4}\right) \\ &= 3 \end{aligned}$$