Sequences and Series

A *sequence* is a set of numbers written in a specific order. For Example:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and a_n is the *nth term*.

Definition: A **sequence** is a function f whose domain is the set of natural numbers. The values $f(1), f(2), f(3), \dots$ are called the **terms** of the sequence, and are frequently written as $a_1, a_2, a_3, \dots$.

A sequence can be written in a list form, when it is clear what the subsequent terms are, with dots following to indicate the sequence continues indefinitely:

$$5, 10, 15, 20, \dots, a_n, \dots$$

or, it can be written as a formula for the *n*th term a_n of the sequence.

$$a_n = 5n$$

Finding the Terms of a Sequence:

Example 1: Find the first five terms and the 100th term of the sequence defined by each formula.

(a) $a_n = 3n - 6$ (b) $b_n = n^2$

(c) $c_n = \dfrac{n+1}{n+2}$ (d) $d_n = \dfrac{(-1)^n}{3^n}$

Step 1: To find the first term a_1, substitute n = 1 into the formula for a_n.

$$a_1 = 3(1) - 6$$
$$a_1 = -3$$

Step 2: To find the second term a_2, substitute $n = 2$ into the formula for a_n.

$$a_2 = 3(2) - 6$$
$$a_2 = 0$$

Example 1 (Continued):

Step 3: Find the subsequent terms in a similar manner. To find the 100^{th} term a_{100}, substitute $n = 100$ into the formula for a_n.

$$a_{100} = 3(100) - 6$$
$$a_{100} = 294$$

Solution:

	nth term	First five terms	100^{th} term
(a)	$3n - 6$	-3, 0, 3, 6, 9	294
(b)	n^2	1, 4, 9, 16, 25	10000
(c)	$\frac{n+1}{n+2}$	$\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$	$\frac{101}{102}$
(d)	$\frac{(-1)^n}{3^n}$	$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}$	$\frac{1}{3^{100}}$

Note: The $(-1)^n$ in Example 1(d) causes the successive terms to be alternately negative and positive.

The following is a definition of a factorial:

If n is a positive integer, then n factorial defined by:
$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n - 1) \cdot n$.

Note: **0!** is defined to be equal to 1.

Factorials follow the same conventions for order of operation as exponents.

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n) \text{ whereas } (2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n \cdot (n + 1) \cdot \ldots \cdot (2n)$$

Example 2: Find the value of **n!** for values of $n = 0 \rightarrow 5$.

Solution:

$0! = 1$
$1! = 1$
$2! = 1 \times 2 = 2$
$3! = 1 \times 2 \times 3 = 6$
$4! = 1 \times 2 \times 3 \times 4 = 24$
$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

From the first example, **n!** is seen to be the product of consecutive integers. It is for this reason that when given factorials in the form of fractions unlike values of n cannot be reduced as if they were common factors i.e. $\mathbf{\frac{10!}{5!} \neq 2!}$ Example 3 will show how to approach and solve such problems.

Example 3: Find $\mathbf{\frac{10!}{5!}}$.

Solution:

Step 1: Analyze.

Since $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ and $10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$ it may be seen that the first five factors are in common. The fraction may then be rewritten as

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!}$$

Step 2: Reduce and solve.

Since the numerator and denominator share the first five values (5!) this value may be reduced from the fraction

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}}$$

leaving a solution of $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$.

Finding the Terms of a Recursive Sequence:

A sequence defined in a way such that the nth term depends on some or all of the preceding terms is called **recursive**.

Example 4: Find the first five terms of the sequence defined recursively by

$$a_1 = 0 \text{ and } a_n = 2a_{n-1} + 1$$

Solution: Because the sequence is recursive, the nth term a_n can be found if the preceding term a_{n-1} is known. Since we are given the first term $a_1 = 0$, we can find the second term, and therefore, then find the third term, and so on.

Step 1: We are given the first term a_1, so now we use it to find the second term a_2. To do this, substitute a_1 into a_n.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2\,a_1 + 1 \\ &= 2(0) + 1 \\ &= 1 \end{aligned}$$

Step 2: To find the third term a_3, substitute a_2 into a_n.

$$\begin{aligned} a_2 &= 1 \\ a_3 &= 2\,a_2 + 1 \\ &= 2(1) + 1 \\ &= 3 \end{aligned}$$

Step 3: Find the subsequent terms in the same manner.

$$\begin{aligned} a_4 &= 2\,a_3 + 1 \\ &= 2(3) + 1 \\ &= 7 \end{aligned}$$

$$\begin{aligned} a_5 &= 2\,a_4 + 1 \\ &= 2(7) + 1 \\ &= 15 \end{aligned}$$

Thus, the first five terms of this sequence are:

$$0, 1, 3, 7, 15$$

Note: In order to find the 100^{th} term of the recursively defined sequence in Example 4, all 99 preceding terms must first be found.

Note: Not all sequences can be defined in the ways discussed above, by a formula or recursively. For example, there is no known formula that produces the sequence of prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

Finding the nth Term of a Sequence:

In order to find the nth term of a sequence by looking at its terms, one must try to find a pattern in the given numbers. For instance, consider a sequence that begins

$$2, 4, 8, 16, \ldots$$

One pattern for this sequence is $2^1, 2^2, 2^3, 2^4, \ldots$. Thus, the sequence could be defined by $a_n = 2^n$. However, this is not the only sequence whose first four terms are 2, 4, 8, 16. Another sequence might be $a_n = 2(2^{n-1})$, or the sequence could be defined recursively as $a_1 = 2$, $a_n = 2a_{n-1}$. In other words, the solution is not unique. But, we only need to find the most obvious sequence whose first few terms agree with the given ones.

Example 5: Find the nth term of a sequence whose first several terms are given.

(a) $2, 4, 6, 8, \ldots$

(b) $\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \ldots$

(c) $1, -3, 5, -7, \ldots$

Solution (a):

Step 1: First, we look for a pattern in the given terms that consists of consecutive integers. One pattern is

$$2(1), 2(2), 2(3), 2(4)$$

Step 2: Since these numbers are 2 times the consecutive integers 1, 2, 3, 4, the sequence we are looking for is defined by

$$a_n = 2n$$

Note: This is the definition for the sequence of the even numbers.

Example 5 (Continued):

Solution (b): Since the given terms are fractions, we will look for a pattern in the numerator, and a pattern in the denominator separately.

Step 1: First, look for a pattern in the numerators of the fractions. We notice the numerators are the even numbers. Even numbers are of the form 2n, as we learned in example 4(a).

Step 2: Now, look for a pattern in the denominators of the fractions. The numbers in the denominators are 3, 9, 27, 81. A pattern here is

$$3^1,\ 3^2,\ 3^3,\ 3^4$$

These numbers are 3 to the powers 1, 2, 3, 4, so the denominator is of the form 3^n.

Step 3: Combining the patterns we found for the numerator and the denominator, we get

$$a_n = \frac{2n}{3^n}$$

Solution (c): If the given terms of a sequence are alternating in sign, look at just the numbers for a pattern. Once the pattern is found, multiplying it by either $(-1)^n$ if the first term is negative, or $(-1)^{n+1}$ if the first term is positive will create the alternating signs effect.

Step 1: Since the given terms are alternating sign, we first look for a pattern in just the numbers, 1, 3, 5, 7. We notice these are the odd numbers. An odd number differs from an even number by one, $2-1$, $4-1$, $6-1$, $8-1$.

Thus, odd numbers are of the form $2n-1$.

Step 2: Now, since the first term is positive, we will multiply $2n-1$ by $(-1)^{n+1}$ to define the sequence. Thus, the sequence we are looking for is defined by

$$a_n = (-1)^{n+1}(2n-1)$$

The Partial Sums of a Sequence:

A *partial sum* is obtained by adding the terms of a sequence.

Definition: For the sequence $a_1, a_2, a_3, a_4, \dots, a_n, \dots$

the **partial sums** are

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ S_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned}$$

S_1 is called the **first partial sum**, S_2 is the **second partial sum**, and so on. S_n is called the **nth partial sum**. The sequence $S_1, S_2, S_3, \dots, S_n, \dots$ is called the **sequence of partial sums**.

Finding the Partial Sums of a Sequence:

Example 6: Find the first four partial sums, and the nth partial sum of the sequence defined by $a_n = \dfrac{1}{3^n}$.

Solution:

Step 1: To find the first partial sum of the sequence, we will calculate the first term of the sequence.

$$a_1 = \frac{1}{3}$$

The first term of the sequence is the first partial sum of the sequence. Thus

$$S_1 = a_1 = \frac{1}{3}$$

Example 6 (Continued):

Step 2: To find the second partial sum, we calculate the first two terms of the sequence.

$$a_1 = \frac{1}{3}\ ,\ a_2 = \frac{1}{9}$$

Then we add the two terms together. Thus

$$S_2 = a_1 + a_2 = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$$

Step 3: Find the third and fourth partial sums in the same manner. First calculate the necessary terms of the sequence, and then add them together to obtain the partial sum.

$$a_1 = \frac{1}{3}\ ,\ a_2 = \frac{1}{9}\ ,\ a_3 = \frac{1}{27}\ ,\ a_4 = \frac{1}{81}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{13}{27}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} = \frac{40}{81}$$

Step 4: Now, to find the *n*th partial sum, we will treat our first four partial sums like the first terms of a sequence. Then, we find the *n*th partial sum in the same way we would find the *n*th term of that sequence. The first four partial sums were

$$\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}$$

The numbers in the denominators, 3, 9, 27, 81, are of the form 3^n, as we found in example 3(b). Notice that the numerator in each partial sum is half of one less than the denominator. That is

$$1 = \frac{1}{2}(3-1) \qquad 13 = \frac{1}{2}(27-1)$$

$$4 = \frac{1}{2}(9-1) \qquad 40 = \frac{1}{2}(81-1)$$

Example 6 (Continued):

Therefore, the numerator is of the form $\frac{1}{2}\left(3^n - 1\right)$, and finally

$$S_n = \frac{\frac{1}{2}\left(3^n - 1\right)}{3^n} = \frac{1}{2}\left(\frac{3^n - 1}{3^n}\right)$$

Sigma Notation:

Given a sequence $a_1, a_2, a_3, a_4, ...$ the sum of the first n terms can be written using **summation notation**, or **sigma notation**. This notation derives its name from the Greek letter $\sum$ (capital sigma, corresponding to our S for "sum"). Sigma notation is used as follows:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + ... + a_n$$

The left side of this expression is read "the sum of a_k from k=1 to k=*n*." The letter k is called the **index of summation**, or the **summation variable**. The idea is to replace k in the expression a_k with the integers 1, 2, 3, … , *n* , and then add the resulting terms together.

Example 7: Find each sum.

(a) $\sum_{k=1}^{5} k$ (b) $\sum_{k=1}^{5} 3$ (c) $\sum_{j=2}^{4} \frac{1}{j}$ (d) $\sum_{m=4}^{7} (-1)^m (m)$

Solution (a): We read this expression as "the sum of k from k=1 to k=5."

Step 1: First, we calculate the individual terms. We do this by replacing the k in a_k with the values 1, 2, 3, 4, 5, to find a_1, a_2, a_3, a_4, a_5.

$$a_k = k \qquad a_3 = 3$$
$$a_1 = 1 \qquad a_4 = 4$$
$$a_2 = 2 \qquad a_5 = 5$$

Step 2: Now, we add the terms together to get the sum.

$$\sum_{k=1}^{5} k = a_1 + a_2 + a_3 + a_4 + a_5$$
$$= 1 + 2 + 3 + 4 + 5 = 15$$

Example 7 (Continued):

Solution (b): We read this expression as "the sum of 3 from k=1 to k=5."

Step 1: First, we calculate the individual terms. In this problem, $a_k = 3$**,** so every term will be 3. Hence,

$$a_k = 3$$
$$a_1 = 3 \qquad a_2 = 3 \qquad a_3 = 3$$
$$a_4 = 3 \qquad a_5 = 3$$

Step 2: Now, we add the terms together to get the sum.

$$\begin{aligned}\sum_{k=1}^{5} 3 &= a_1 + a_2 + a_3 + a_4 + a_5 \\ &= 3 + 3 + 3 + 3 + 3 \\ &= 15\end{aligned}$$

Solution (c): We read this expression as "the sum of $\frac{1}{j}$ from j=2 to j=4."

Step 1: In this problem notice the index of summation will start at 2, instead of 1, and end at 4. So, first we will calculate the terms of a_j from a_2 to a_4.

$$a_j = \frac{1}{j}$$
$$a_2 = \frac{1}{2} \qquad a_3 = \frac{1}{3} \qquad a_4 = \frac{1}{4}$$

Step 2: Now, we add the terms together to get the sum.

$$\begin{aligned}\sum_{j=2}^{4} \frac{1}{j} &= a_2 + a_3 + a_4 \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\ &= \frac{26}{24} = \frac{13}{12}\end{aligned}$$

Example 7 (Continued):

Solution (d): We read this expression as "the sum of $(-1)^m(m)$ from m=4 to m=7."

Step 1: As in example 1(d), the $(-1)^m$ in the expression will cause successive terms to alternate in sign. Calculate the terms of a_m from a_4 to a_7.

$$a_m = (-1)^m(m)$$

$$a_4 = (-1)^4(4) = 4 \qquad a_6 = (-1)^6(6) = 6$$

$$a_5 = (-1)^5(5) = -5 \qquad a_7 = (-1)^7(7) = -7$$

Step 2: Now, add the terms together to get the sum.

$$\begin{aligned}\sum_{m=4}^{7}(-1)^m(m) &= a_4 + a_5 + a_6 + a_7 \\ &= 4 + (-5) + 6 + (-7) \\ &= -2\end{aligned}$$

Example 8: Write each sum using summation notation.

(a) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$ (b) $-1+2-3+4-5+6$

Solution (a):

Step 1: First, find a pattern in the given terms that consists of consecutive integers. Since the terms here are fractions, look for a pattern in the numerators and the denominators separately. For this problem, the numerators are all 1. In the denominators, a pattern would be 2, 3, 4, 5, 6. To write the pattern in general form, replace the consecutive integers with k. Hence,

$$a_k = \frac{1}{k}$$

Step 2: Next, find where the index of summation begins and ends. Since the first consecutive integer in the pattern is 2 and the last is 6, the index of summation will begin at 2 and end at 6.

Step 3: Finally, write the sum using summation notation.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6} = \sum_{k=2}^{6}\frac{1}{k}$$

Example 8 (Continued):

Solution (b):

Step 1: Notice in this problem the terms alternate in sign. To find a pattern, we will look at just the numbers. Here, a pattern that consists of consecutive integers is 1, 2, 3, 4, 5, 6. Thus, the general form is k. Also, similar to example 4(c), multiplying the general form by either $(-1)^k$ if the first term is negative, or $(-1)^{k+1}$ if the first term is positive will create the alternating signs effect. Hence,

$$a_k = (-1)^k k$$

Step 2: Since the consecutive integers in the pattern begin at 1 and end at 6, the index of summation will go from 1 to 6. Therefore, the sum written in summation form is

$$\sum_{k=1}^{6} (-1)^k k$$

The following properties are consequences of properties of the real numbers.

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then for every positive integer n and any real number c, the following properties hold.

1. $\sum_{k=1}^{n} \left(a_k + b_k\right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ **2.** $\sum_{k=1}^{n} \left(a_k - b_k\right) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

3. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

Example 9: Use the properties of sums to write $\sum_{i=1}^{6} \left(i^2 + i\right)$ as a sum of two summations.

Solution:

Step 1: Since the given expression contains a +, we will use the first property of sums. According to the property, the new summations will have the same index of summation as the original one. Also, the new summations will each contain one term of the original one. Hence,

$$\sum_{i=1}^{6} \left(i^2 + i\right) = \sum_{i=1}^{6} i^2 + \sum_{i=1}^{6} i$$