COMPLEX NUMBERS

By definition a complex number is one which can be expressed in the form of $a + bi$. In the relationship of numbers, all real numbers are a subset of the set of complex numbers. The number $i$ is used to solve certain equations that cannot be solved using the set of real numbers. In using $i$ it is understood that $i^2 = -1$, that is, $i$ is a number whose square is $-1$. Keeping in mind that solutions involving complex numbers must be in the form of $a + bi$ the following examples will demonstrate how to perform some basic operations involving complex numbers.

Example 1. Add.

a.) \((2 + 3i) + (6 + 4i)\)

Step 1. Group like terms.
\[
(2 + 3i) + (6 + 4i) = 2 + 3i + 6 + 4i = (2 + 6) + (3i + 4i)
\]

Step 2. Factor the $i$ from its group.
\[
(2 + 6) + (3i + 4i) = 8 + (3 + 4)i = 8 + 7i
\]

b.) \(5 + (9 - 3i)\)

Step 1. Group like terms. (Note 5 may be expressed as $5 + 0i$)
\[
5 + (9 - 3i) = 5 + 0i + 9 - 3i = (5 + 9) + (0i - 3i)
\]

Step 2. Factor the $i$ from its group.
\[
(5 + 9) + (0i - 3i) = 14 + (0 - 3)i = 14 + (-3)i \text{ or } 14 - 3i
Example 2. Subtract.

a.) \((6 + 5i) - (3 + 2i)\)

Step 1. Group like terms.
\[(6 + 5i) - (3 + 2i) = 6 + 5i - 3 - 2i\]
\[= 6 + 5i + (-3) + (-2i)\]
\[= (6 + (-3)) + (5i + (-2i))\]
\[= (6 - 3) + (5i - 2i)\]

Step 2. Factor the \(i\) from its group.
\[(6 - 3) + (5i - 2i) = 3 + (5 - 2)i\]
\[= 3 + 3i\]

b.) \((7 - 3i) - (8 - 6i)\)

Step 1. Group like terms.
\[(7 - 3i) - (8 - 6i) = 7 - 3i - 8 + 6i\]
\[= 7 + (-3i) + (-8) + 6i\]
\[= (7 + (-8)) + ((-3i) + 6i)\]

Step 2. Factor the \(i\) from its group.
\[(7 + (-8)) + ((-3i) + 6i) = -1 + (-3 + 6)i\]
\[= -1 + 3i\]

Dealing with complex numbers that one tries to find a product or quotient of, demands that one understand that since \(i^2 = -1\), \(i = \sqrt{-1}\). The definition \(\sqrt{-b} = i\sqrt{b}\) is used to resolve the square root of a negative radicand. Note that the resolved term is \(i\sqrt{b}\) instead of \(\sqrt{b}\ i\) so that it will not be mistakenly written as \(\sqrt{bi}\).

Example 3. Simplify.

a.) \(\sqrt{-100} = \sqrt{100} \sqrt{-1} = \sqrt{100} \ i = 10i\)

b.) \(\sqrt{-2} = \sqrt{2} \sqrt{-1} = \sqrt{2} \ i = i\sqrt{2}\)
Example 4. Multiply.

a.) \[ \sqrt{-3} \sqrt{-7} = i \sqrt{3} \ i \sqrt{7} = i^2 \sqrt{21} = (-1) \sqrt{21} = -\sqrt{21} \]

b.) \[ \sqrt{-2} \sqrt{-8} = i \sqrt{2} \ i \sqrt{8} = i^2 \sqrt{16} = (-1)(4) = -4 \]

Example 5. Simplify.

a.) \[ (3 + 5i) (4 - 2i) = 12 - 6i + 20i - 10i^2 \]
   \[ = 12 + 14i - [10 (-1)] \]
   \[ = 12 + 14i - [-10] \]
   \[ = 12 + 14i + 10 \]
   \[ = 22 + 14i \]

b.) \[ (2 + 3i) (1 - 5i) = 2 - 10i + 3i - 15i^2 \]
   \[ = 2 - 7i - [15 (-1)] \]
   \[ = 2 - 7i - [-15] \]
   \[ = 2 - 7i + 15 \]
   \[ = 17 - 7i \text{ or } 17 + (-7i) \]

Note that when \(a + bi\) is multiplied by its conjugate, \(a - bi\), the product is \(a^2 + b^2\).

Example 6. Find the quotient.

a.) \[ \frac{4 - 3i}{5 + 2i} \]

Step 1. Multiply the problem by its conjugate. (This is done to resolve the \(i\) values in the denominator.)

\[ \frac{4 - 3i}{5 + 2i} \left( \frac{5 - 2i}{5 - 2i} \right) = \frac{20 - 8i - 15i + 6i^2}{5^2 + 2^2} \]

\[ = \frac{20 - 23i + 6i^2}{25 + 4} \]
Step 2. Simplify.

\[
\frac{20 - 23i + 6i^2}{25 + 4} = \frac{20 - 23i + [(6)(-1)]}{29} = \frac{20 - 23i + (-6)}{29} = \frac{14 - 23i}{29} = \frac{14}{29} - \frac{23i}{29} \text{ or } \frac{14}{29} + \left(-\frac{23i}{29}\right)
\]

b.) \[
\frac{1 + i}{i}
\]

Step 1. Multiply the problem by its conjugate.

\[
\frac{1 + i}{i} \left(\frac{-i}{-i}\right) = \frac{-i - i^2}{-i^2}
\]

Step 2. Simplify.

\[
\frac{-i - i^2}{-i^2} = \frac{-i - (-1)}{(-1)} = \frac{-i + 1}{1} = (-i) + 1
\]

\[
= 1 + (-i)
\]