RATIONAL EXPONENTS

There are two definitions that must be understood to work the problems of this section.

Definition 1. **If b is a real number, n is a positive integer greater than 1, and $\sqrt[n]{b}$ exists, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.**

Example 1. **Simplify.**

a.) $3^{\frac{1}{3}} = \sqrt[3]{3}$

b.) $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

c.) $\left(\frac{25}{81}\right)^{\frac{1}{2}} = \left(\frac{25^{\frac{1}{2}}}{81^{\frac{1}{2}}}\right) = \frac{\sqrt{25}}{\sqrt{81}} = \frac{5}{9}$

Definition 2. **If $\frac{m}{n}$ is a rational number, where n is a positive integer greater than 1, and b is a real number such that $\sqrt[n]{b}$ exists, then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.**

Example 2. **Simplify.**

a.) $\sqrt[9]{5^6} = 5^{\frac{6}{9}} = 5^{\frac{2}{3}} = \sqrt[3]{5^2} = \sqrt[3]{25}$

b.) $\sqrt[4]{m^2} = m^{\frac{2}{4}} = m^{\frac{1}{2}} = \sqrt{m}$

Note: Sometimes it is easier to work a problem if the radicand is also prime factored.

c.) Simplify $\sqrt[4]{16^2}$ **.**

$\sqrt[4]{16^2}$	**or**	$\sqrt[4]{16^2}$	**or**	$\sqrt[4]{16^2}$
$\sqrt[4]{256}$	**or**	$\left(\sqrt[4]{16}\right)^2$	**or**	$\sqrt[4]{\left(2^4\right)^2}$
4	**or**	$16^{\frac{1}{2}}$	**or**	$2^{\frac{8}{4}}$
4	**or**	$\sqrt{16}$	**or**	2^2
4	**or**	**4**	**or**	**4**

As you can see, the first method requires you to know that the fourth root of 256 is four. The second and third methods let you factor to a root that you are more likely to know.

The definitions allow one to simplify the product of two or more radicals having different index numbers.

Example 3. Solve $\left(\sqrt{7}\right)\left(\sqrt[3]{2}\right)$ **.**

Step 1. Convert the terms from radical to exponential form.

$\left(\sqrt{7}\right) = 7^{\frac{1}{2}}$ $\quad\quad$ $\left(\sqrt[3]{2}\right) = 2^{\frac{1}{3}}$

Step 2. Find an LCD for the fractional exponents.

In this example the LCD = 6.

$7^{\left(\frac{1}{2}\right)\left(\frac{3}{3}\right)} = 7^{\frac{3}{6}} = \left(\sqrt[6]{7^3}\right)$ $\quad\quad$ $2^{\left(\frac{1}{3}\right)\left(\frac{2}{2}\right)} = 2^{\frac{2}{6}} = \left(\sqrt[6]{2^2}\right)$

Step 3. Simplify.

$$\left(\sqrt{7}\right)\left(\sqrt[3]{2}\right) =$$

$$\left(\sqrt[6]{7^3}\right)\left(\sqrt[6]{2^2}\right) =$$

$$\left(\sqrt[6]{343}\right)\left(\sqrt[6]{4}\right) =$$

$$\left(\sqrt[6]{1372}\right)$$

To review, the exponential rules that you should know at this point are:

1.) $b^x * b^y = b^{x+y}$

2.) $b^x \div b^y = b^{x-y}$

3.) $(b^x)^y = b^{xy}$

4.) $b^0 = 1 ;\ b \neq 0$

5.) $b^{-x} = \frac{1}{b^x}$

Example 4. **Solve** $\left(\frac{15x^3y^4z^{-4}}{25x^4y^3z^{-5}}\right)^3\left(\frac{12a^{-5}b^{-2}c^2}{9a^3d^2x^{-2}z^3}\right)^{-2}$

Step 1. Prime factor within the groups.

$$\left(\frac{15x^3y^4z^{-4}}{25x^4y^3z^{-5}}\right)^3\left(\frac{12a^{-5}b^{-2}c^2}{9a^3d^2x^{-2}z^3}\right)^{-2}$$

$$\left(\frac{3^1 5^1 x^3y^4z^{-4}}{5^2x^4y^3z^{-5}}\right)^3\left(\frac{2^2 3^1 a^{-5}b^{-2}c^2}{3^2a^3d^2x^{-2}z^3}\right)^{-2}$$

Example 4 (Continued):

Step 2. Apply the outer exponentiation to the groups using the rule $(b^x)^y = b^{xy}$**.**

$$\left(\frac{3^1 5^1 x^3 y^4 z^{-4}}{5^2 x^4 y^3 z^{-5}}\right)^3 \quad \left(\frac{2^2 3^1 a^{-5} b^{-2} c^2}{3^2 a^3 d^2 x^{-2} z^3}\right)^{-2}$$

$$\left(\frac{3^3 5^3 x^9 y^{12} z^{-12}}{5^6 x^{12} y^9 z^{-15}}\right) \quad \left(\frac{2^{-4} 3^{-2} a^{10} b^4 c^{-4}}{3^{-4} a^{-6} d^{-4} x^4 z^{-6}}\right)$$

Step 3. Multiply across the terms using the rule $b^x * b^y = b^{x+y}$**.**

$$\frac{3^3 5^3 x^9 y^{12} z^{-12} 2^{-4} 3^{-2} a^{10} b^4 c^{-4}}{5^6 x^{12} y^9 z^{-15} 3^{-4} a^{-6} d^{-4} x^4 z^{-6}}$$

$$\frac{2^{-4} 3^1 5^3 a^{10} b^4 c^{-4} x^9 y^{12} z^{-12}}{3^{-4} 5^6 a^{-6} d^{-4} x^{16} y^9 z^{-21}}$$

Step 4. Insert missing bases into the problem using the rule $b^0 = 1\ ;\ b \neq 0$ **. In this problem d is missing from the numerator while in the denominator 2, b, and c are missing.**

$$\frac{2^{-4} 3^1 5^3 a^{10} b^4 c^{-4} d^0 x^9 y^{12} z^{-12}}{2^0 3^{-4} 5^6 a^{-6} b^0 c^0 d^{-4} x^{16} y^9 z^{-21}}$$

Example 4 (Continued):

Step 5. Placement.

Wherever the exponent is the most positive for a base is where that base will finish in the solution. The rule that deals with the exponent of the like term is $b^{-x} = \frac{1}{b^x}$. In this example the bases of the numerator are: 3, a, b, d, y, z. The bases of the denominator are: 2, 5, c, x. Applying the rule the problem is written as follows:

$$\frac{3^{1-(-4)}a^{10-(-6)}b^{4-0}d^{0-(-4)}y^{12-9}z^{-12-(-21)}}{2^{0-(-4)}5^{6-3}c^{0+4}x^{16-9}}$$

Step 6. Simplify.

$$\frac{3^{1+4}a^{10+6}b^{4-0}d^{0+4}y^{12-9}z^{-12+21}}{2^{0+4}5^{6-3}c^{0+4}x^{16-9}}$$

$$\frac{3^5a^{16}b^4d^4y^3z^9}{2^45^3c^4x^4}$$

This is the solution because there is only one of any base and all of the exponents are positive.